century engineers were aware of this but regretted it and tried to build structures (which quite often fell down) making use of such theory as was available to them. English engineers, who were also aware of it, were usually indifferent to 'theory' and they built the structures of the Industrial Revolution by rule-of-thumb 'practical' methods. These structures probably fell down nearly, but not quite, as often.

Chapter 3 The invention of stress and strain

– or Baron Cauchy and the decipherment of Young's modulus

What would life be without arithmetic, but a scene of horrors?

Rev. Sydney Smith, letter to a young lady, 22 July 1835

Apart from Newton and the prejudices of the eighteenth century, the main reason why the science of elasticity got stuck for so long was that the few scientists who did study it tried to deal with forces and deflections by considering the structure as a whole – as Hooke had done – rather than by analysing the forces and extensions which could be shown to exist at any given *point within* the material. All through the eighteenth century and well into the nineteenth, very clever men, such as Leonhard Euler (1707–83) and Thomas Young (1773–1829), performed what must appear to the modern engineer to be the most incredible intellectual contortions in their attempts to solve what now seem to us to be quite straightforward problems.

The concept of the elastic conditions at a specified point inside a material is the concept of stress and strain. These ideas were first put forward in a generalized form by Augustin Cauchy (1789–1857) in a paper to the French Academy of Sciences in 1822. This paper was perhaps the most important event in the history of elasticity since Hooke. After this, that science showed promise of becoming a practical tool for engineers rather than a happy hunting-ground for a few somewhat eccentric philosophers. From his portrait, painted at about this time, Cauchy looks rather a pert young man, but he was undoubtedly an applied mathematician of great ability.

When, eventually, English nineteenth-century engineers bothered to read what Cauchy had said on the subject, they found that, not only were the basic concepts of stress and strain really quite easy to understand, but, once they had been understood, the whole study of structures was much simplified. Nowadays these ideas can be understood by anybody,* and it is hard to account for the bewildered and even resentful attitude which is sometimes taken up by laymen when 'stresses and strains' are mentioned. I once had a research student with a nice new degree in zoology who was so upset by the whole idea of stress and strain that she ran away from the university and hid herself. I still do not see why.

Stress – which is not to be confused with strain

As it happened, Galileo himself very nearly stumbled upon the idea of stress. In the *Two New Sciences*, the book he wrote in his old age at Arcetri, he states very clearly that, other things being equal, a rod which is pulled in tension has a strength which is proportional to its cross-sectional area. Thus, if a rod of two square centimetres cross-section breaks at a pull of 1,000 kilograms, then one of four square centimetres cross-section will need a pull of 2,000 kilograms force in order to break it, and so on. That it should have taken nearly two hundred years to divide the breaking load by the area of the fracture surface, so as to get what we should now call a 'breaking stress' (in this case 500 kilograms per square centimetre) which might be applied to all similar rods made from the same material almost passes belief.

Cauchy perceived that this idea of stress can be used, not only to predict when a material will break, but also to describe the state of affairs at any point inside a solid in a much more general kind of way. In other words the 'stress' in a solid is rather like the 'pressure' in a liquid or a gas. It is a measure of how hard the atoms and molecules which make up the material are being pushed together or pulled apart as a result of external forces.

Thus, to say 'The stress at that point in this piece of steel is 500 kilograms per square centimetre' is no more obscure or mysterious than to say 'The pressure of the air in the tyres of my car is 2 kilograms per square centimetre – or 28 pounds per square

*Except, apparently, the *Oxford Dictionary*. The words are used, of course, in casual conversation to describe the mental state of people and as if they meant the same thing. In physical science the meanings of the two words are quite clear and distinct.

inch.' However, although the concepts of pressure and stress are fairly closely comparable, we have to bear in mind that pressure acts in all three directions within a fluid while the stress in a solid is often a directional or one-dimensional affair. Or, at any rate, so we shall consider it for the present.

Numerically, the stress in any direction at a given point in a material is simply the force or load which happens to be acting in that direction at the point, divided by the area on which the force acts.* If we call the stress at a certain point s, then

$$\text{Stress} = s = \frac{\text{load}}{\text{area}} = \frac{P}{A}$$

where P = load or force and A is the area over which the force P can be considered as acting.

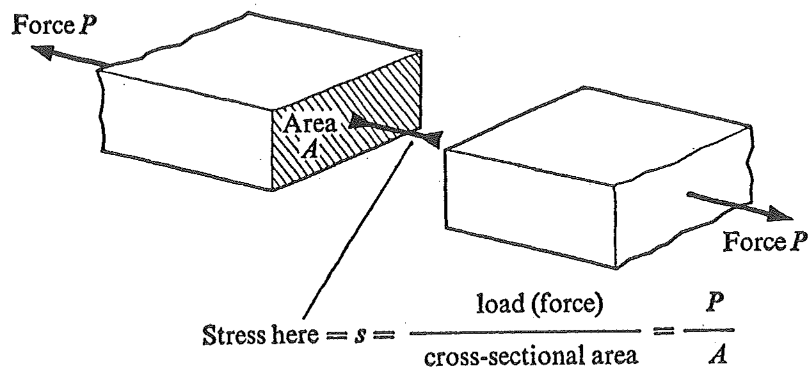


Figure 1. Stress in a bar under tension. (Compressive stress is exactly analogous.)

To revert to our brick, which we left in the last chapter hanging from its string. If the brick weighs 5 kilograms and the string has a cross-section of 2 square millimetres, then the brick pulls on the string with a force of 5 kilograms, and the stress in the string will be:

*How can a 'point' have an 'area'? Consider the analogy of speed: we express speed as the distance covered in a certain length of time, e.g. miles per hour, although we are concerned usually with the speed at any given – infinitely brief – moment.

$$s = \frac{\text{load}}{\text{area}} = \frac{P}{A} = \frac{5 \text{ kilograms force}}{2 \text{ square millimetres}}$$
$$= 2{\cdot}5 \text{ kilograms force per square millimetre}$$

or, if we prefer it, 250 kilograms force per square centimetre or kgf/cm^2.

Units of stress

This raises the vexed question of units of stress. Stress can be expressed in any units of force divided by any units of area – and it frequently is. To reduce the amount of confusion we shall stick to the following units in this book.

MEGANEWTONS PER SQUARE METRE: MN/m^2. This is the SI unit. As most people know, the SI (System International) habit is to make the unit of force the Newton.
1·0 Newton = 0·102 kilograms force = 0·225 pounds force (roughly the weight of one apple).
1 Meganewton = one million Newtons, which is almost exactly 100 tons force.

POUNDS (FORCE) PER SQUARE INCH: p.s.i. This is the traditional unit in English-speaking countries, and it is still very widely used by engineers, especially in America. It is also in common use in a great many tables and reference books.

KILOGRAMS (FORCE) PER SQUARE CENTIMETRE: kgf/cm^2 (sometimes kg/cm^2). This is the unit in common use in Continental countries, including Communist ones.

FOR CONVERSION

1 MN/m^2 = 10·2 kgf/cm^2 = 146 p.s.i.
1 p.s.i. = 0·00685 MN/m^2 = 0·07 kgf/cm^2
1 kgf/cm^2 = 0·098 MN/m^2 = 14·2 p.s.i.

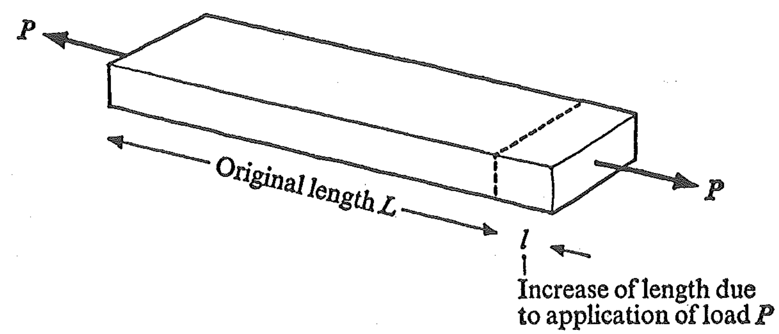
Thus the stress in our piece of string, which we found to be 250 kgf/cm^2, is also equal to 24·5 MN/m^2 or 3,600 p.s.i. Since the

calculation of stresses is not usually a very accurate business, there is no sense in fussing too much about very exact conversion factors.

It is worth repeating that it is important to realize that the stress in a material, like the pressure in a fluid, is a condition which exists at a *point* and it is not especially associated with any particular cross-sectional area, such as a square inch or a square centimetre or a square metre.

Strain – which is not the same thing as stress

Just as stress tells us how *hard* – that is, with how much force – the atoms at any point in a solid are being pulled apart, so strain tells us how *far* they are being pulled apart – that is, by what proportion the bonds between the atoms are stretched.



$$\text{Strain} = \frac{\text{increase of length}}{\text{original length}} = \frac{l}{L} = e$$

Figure 2. Strain in a bar under tension. (Compressive strain is exactly analogous.)

Thus, if a rod which has an original length L is caused to stretch by an amount l by the action of a force on it, then the *strain*, or proportionate change of length, in the rod will be e, let us say, such that:

$$e = \frac{l}{L}$$

To return to our string, if the original length of the string was, say, 2 metres (or 200 cm), and the weight of the brick causes it to stretch by 1 centimetre, then the strain in the string is:

$$e = \frac{l}{L} = \frac{1}{200} = 0{\cdot}005 \text{ or } 0{\cdot}5\%$$

Engineering strains are usually quite small, and so engineers very often express strains as percentages, which reduces the opportunities for confusion with noughts and decimal points.

Like stress, strain is not associated with any particular length or cross-section or shape of material. It is also a condition at a point. *Again, since we calculate strain by dividing one length by another length – i.e. the extension by the original length – strain is a ratio, which is to say a number, and it has no units, SI, British or anything else.* All this applies just as much in compression, of course, as it does in tension.

Young's modulus – or how stiff is this material?

As we have said, Hooke's law in its original form, though edifying, was the result of a rather inglorious muddle between the properties of materials and the behaviour of structures. This muddle arose mainly from the lack of the concepts of stress and strain, but we also have to bear in mind the difficulties which would have existed in the past in connection with testing materials.

Nowadays, when we want to test a material – as distinct from a

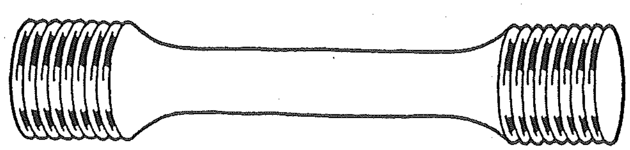


Figure 3. A typical tensile test-piece.

structure – we generally make what is called a 'test-piece' from it. The shapes of test-pieces may vary a good deal but usually they have a parallel stem, on which measurements can be made, and are provided with thickened ends by which they can be attached

to the testing machine. An ordinary metal test-piece often looks like Figure 3.

Testing machines also vary a good deal in size and in design, but basically they are all mechanical devices for applying a measured load in tension or in compression.

The stress in the stem of the test-piece is obtained merely by dividing the load recorded at each stage on the dial of the machine by the area of its cross-section. The extension of the stem of the test-piece under load – and therefore the strain in the material – is usually measured by means of a sensitive device called an extensometer, which is clamped to two points on the stem.

With equipment of this kind it is generally quite easy to measure the stress and the strain which occur within a specimen of a

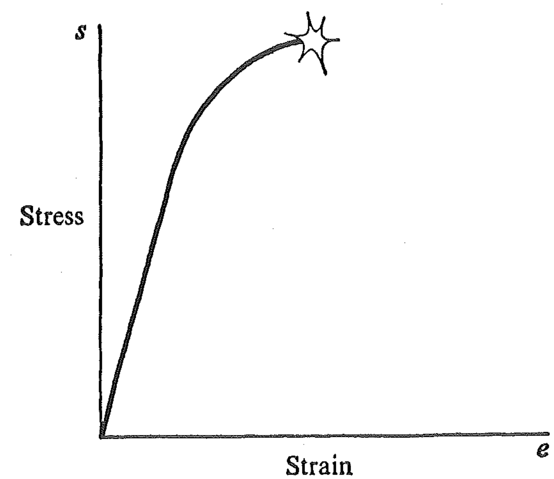


Figure 4. A typical 'stress-strain diagram'.

material as we increase the load upon it. The relationship between stress and strain for that material is given by the graph of stress plotted against strain which we call the 'stress-strain diagram'. This stress-strain diagram, which may look something like Figure 4, is very characteristic of any given material, and its shape is usually unaffected by the size of the test-piece which happens to have been used.

When we come to plot the stress-strain diagram for metals and for a number of other common solids we are very apt to find that, at least for moderate stresses, the graph is a straight line. When this is so we speak of the *material* as 'obeying Hooke's law' or sometimes of a 'Hookean material'.

What we also find, however, is that the *slope* of the straight part of the graph varies greatly for different materials (Figure 5). It is

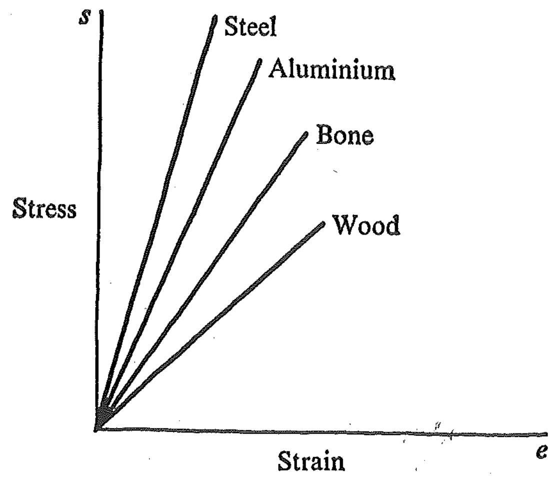


Figure 5. The slope of the straight part of the stress-strain diagram is characteristic of each different material. *E*, the Young's modulus of elasticity, represents this slope.

clear that the slope of the stress-strain diagram measures how readily each material strains elastically under a given stress. In other words it is a measure of the elastic stiffness or floppiness of a given solid.

For any given material which obeys Hooke's law, the slope of the graph or the ratio of stress to strain will be constant. Thus for any particular material

$$\frac{\text{stress}}{\text{strain}} = \frac{s}{e} = \text{Young's modulus of elasticity, which we call } E$$
$$= \text{constant for that material}$$

Young's modulus is sometimes called 'the elastic modulus' and sometimes '*E*', and is quite often spoken of as 'stiffness' in

ordinary technical conversation. The word 'modulus', by the way, is Latin for 'a little measure'.

Our string, it may be remembered, was strained 0·5 per cent or 0·005 by the weight of the brick, which imposed a stress of 24·5 MN/m² or 3,600 p.s.i. The Young's modulus of the string is therefore

$$\frac{\text{stress}}{\text{strain}} = \frac{24{\cdot}5}{0{\cdot}005} = 4{,}900\ \text{MN/m}^2$$
$$= 720{,}000\ \text{p.s.i.}$$

Units of stiffness or Young's modulus

Since we are dividing a stress by a fraction, which is to say a number, which has no dimensions, Young's modulus has the same dimensions as a stress and is expressed in stress units, that is to say MN/m², p.s.i. or kgf/cm². Since, however, Young's modulus may be regarded as that stress which would double the length of the material (i.e. the stress at 100 per cent strain) – if the material did not break first – the numbers involved are often large, and some people find them difficult to visualize.

Practical values of Young's modulus

The Young's moduli of a number of common biological and engineering materials are given in Table 1. Starting from the cuticle of the pregnant locust (which is low, but not very exceptionally low, for biological materials; the cuticle of the male locust and of the virgin female locust is a lot stiffer, by the way) the Young's moduli are arranged in ascending order all the way to diamond. It will be seen that the range of stiffness varies by about 6,000,000 to one. Which is a lot. We shall discuss why this should be so in Chapter 8.

It may be noticed that a good many common soft biological materials do not occur in this table. This is because their elastic behaviour does not obey Hooke's law, even approximately, so that it is really impossible to define a Young's modulus, at any

TABLE 1

Approximate Young's moduli of various solids

Material	Young's modulus (E) p.s.i.	Young's modulus (E) MN/m²
Soft cuticle of pregnant locust*	30	0·2
Rubber	1,000	7
Shell membrane of egg	1,100	8
Human cartilage	3,500	24
Human tendon	80,000	600
Wallboard	200,000	1,400
Unreinforced plastics, polythene, nylon	200,000	1,400
Plywood	1,000,000	7,000
Wood (along grain)	2,000,000	14,000
Fresh bone	3,000,000	21,000
Magnesium metal	6,000,000	42,000
Ordinary glasses	10,000,000	70,000
Aluminium alloys	10,000,000	70,000
Brasses and bronzes	17,000,000	120,000
Iron and steel	30,000,000	210,000
Aluminium oxide (sapphire)	60,000,000	420,000
Diamond	170,000,000	1,200,000

* By courtesy of Dr Julian Vincent, Department of Zoology, University of Reading.

rate in the terms we have been talking about. We shall come back to this sort of elasticity later on.

Young's modulus is nowadays regarded as a pretty fundamental concept; it thoroughly pervades engineering and materials science and is beginning to invade biology. Yet it took all of the first half of the nineteenth century for the penny to drop in the minds of engineers. This was partly due to sheer conservatism and partly due to the late arrival of any workable concept of stress and strain.

Given these ideas, few things are simpler or more obvious than Young's modulus; without them, the whole affair must have seemed impossibly difficult. Young, who was to play an important part in the decipherment of Egyptian hieroglyphics and who had one of the finest brains of his generation, obviously had a very severe intellectual struggle.

Working around the year 1800, he had to approach the problem by a route quite different from that which we have just used, and he considered the question in terms of what we should now call the 'specific modulus', that is, by how much a column of a material might be expected to shorten under its own weight. Young's own definition of his modulus, published in 1807, is as follows: 'The modulus of the elasticity of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a certain degree of compression as the length of the substance is to the diminution of its length.'*

After which, Egyptian hieroglyphics must have appeared simple.

It was said of Young by one of his contemporaries that 'His words were not those in familiar use, and the arrangement of his ideas seldom the same as those he conversed with. He was therefore worse calculated than any man I ever knew for the communication of knowledge.' All the same we have to realize that Young was wrestling with an idea that was scarcely capable of expression without the concepts of stress and strain, which did not come into use until fifteen or twenty years later. The modern definition of Young's modulus ($E =$ stress/strain) was given in 1826 – three years before Young died – by the French engineer Navier (1785–1836). As the inventor of stress and strain, Cauchy was eventually made a baron by the French government. He seems to have deserved it.

Strength

It is necessary to avoid confusion between the strength of a structure and the strength of a material. The strength of a *structure* is simply the *load* (in pounds force or Newtons or kilograms force) which will just break the structure. This figure is known as the 'breaking load', and it naturally applies only to some individual, specific structure.

* 'Though science is much respected by their Lordships and your paper is much esteemed, it is too learned . . . in short it is not understood' (Admiralty letter to Young).

The strength of a *material* is the *stress* (in p.s.i. or MN/m^2 or kgf/cm^2) required to break a piece of the material itself. It will generally be the same for all specimens of any given solid. We are most often concerned with the tensile strength of materials, which is sometimes called the 'ultimate tensile stress' or U.T.S. This is usually determined by breaking small test-pieces in a testing machine. Naturally, the object of many strength calculations is to

TABLE 2

Approximate tensile strengths of various solids

Material	Tensile strength p.s.i.	Tensile strength MN/m^2
Non-metals		
Muscle tissue (fresh but dead)	15	0·1
Bladder wall („ „ „)	34	0·2
Stomach wall („ „ „)	62	0·4
Intestine („ „ „)	70	0·5
Artery wall („ „ „)	240	1·7
Cartilage („ „ „)	430	3·0
Cement and concrete	600	4·1
Ordinary brick	800	5·5
Fresh skin	1,500	10·3
Tanned leather	6,000	41·1
Fresh tendon	12,000	82
Hemp rope	12,000	82
Wood (air dry): along grain	15,000	103
across grain	500	3·5
Fresh bone	16,000	110
Ordinary glass	5,000–25,000	35–175
Human hair	28,000	192
Spider's web	35,000	240
Good ceramics	5,000–50,000	35–350
Silk	50,000	350
Cotton fibre	50,000	350
Catgut	50,000	350
Flax	100,000	700
Fibreglass plastics	50,000–150,000	350–1,050
Carbon-fibre plastics	50,000–150,000	350–1,050
Nylon thread	150,000	1,050

Material	Tensile strength p.s.i.	Tensile strength MN/m^2
Metals		
STEELS		
Steel piano wire (very brittle)	450,000	3,100
High tensile engineering steel	225,000	1,550
Commercial mild steel	60,000	400
WROUGHT IRON		
Traditional	15,000–40,000	100–300
CAST IRON		
Traditional (very brittle)	10,000–20,000	70–140
Modern	20,000–40,000	140–300
OTHER METALS		
Aluminium: cast	10,000	70
wrought alloys	20,000–80,000	140–600
Copper	20,000	140
Brasses	18,000–60,000	120–400
Bronzes	15,000–80,000	100–600
Magnesium alloys	30,000–40,000	200–300
Titanium alloys	100,000–200,000	700–1,400

predict the strength of a structure from the known strength of its material.

The tensile strengths of a good many materials are given in Table 2. As with stiffness, it will be seen that the range of strengths in both biological and engineering solids is very wide indeed. For instance, the contrast between the weakness of muscle and the strength of tendon is striking, and this accounts for the very different cross-sections of muscles and their equivalent tendons. Thus the thick and sometimes bulging muscle in our calves transmits its tension to the bone of our heel, so that we can walk and jump, by means of the Achilles or calcaneal tendon, which, although it is pencil-thin, is generally quite adequate for the job. Again, we can see why engineers are unwise to put tensile forces on concrete unless that weak material is sufficiently reinforced with strong steel rods.

The strong metals are rather stronger, on the whole, than the strong non-metals. However, nearly all metals are considerably denser than most biological materials (steel has a specific gravity of 7·8, most zoological tissues about 1·1). Thus, strength for weight, metals are not too impressive when compared with plants and animals.

We might now sum up what has been said in this chapter:

$$\textit{Stress} = \frac{\text{load}}{\text{area}}$$

It expresses how *hard* (i.e. with how much force) the atoms at a point within a solid are being pulled apart or pushed together by a load.

$$\textit{Strain} = \frac{\text{extension under load}}{\text{original length}}$$

It expresses how *far* the atoms at a point within a solid are being dragged apart or pushed together.
Stress is not the same thing as strain.
Strength. By the *strength* of a material we usually mean that stress which is needed to break it.

$$\textit{Young's modulus} = \frac{\text{stress}}{\text{strain}} = E.$$

It expresses how stiff or how floppy a material is.
Strength is not the same thing as stiffness.

To quote from *The New Science of Strong Materials*: 'A biscuit is stiff but weak, steel is stiff and strong, nylon is flexible (low E) and strong, raspberry jelly is flexible (low E) and weak. The two properties together describe a solid about as well as you can reasonably expect two figures to do.'

In case you should ever have felt any trace of doubt or confusion on these points, it might be of some comfort to know that, not so long ago, I spent a whole evening in Cambridge trying to explain to two scientists of really shattering eminence and world-

wide fame the basic difference between stress and strain and strength and stiffness in connection with a very expensive project about which they were proposing to advise the government. I am still uncertain how far I was successful.